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1D heat transfer through a flat plate submitted to step changes in heat transfer coefficient

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Abstract

The thermal response of an infinite flat plate is considered, when the heat transfer coefficients on both the exposed surfaces undergo step changes. The configuration is a simplified model for the heat transfer through the separating wall in the Isochoric Counter-Current Heat Exchanger. The step change in the second surface appears with a time delay with respect to the first. The plate temperature, surface heat fluxes and accumulated energy perturbations are evaluated for both the thermally thick and thin cases and the corresponding results are compared. The results show a significant influence of both the delay time and the Biot numbers perturbations. It is shown that for a specific combination of Biot number magnitudes the plate is brought suddenly into a steady state condition and as soon as the second step appears. In addition, an inner point may be defined where the final steady state temperature perturbation becomes zero, separating the plate into two oppositely thermally stressed regions.

Keywords: Isochoric Counter-Current Heat Exchanger; Heat transfer coefficient; Step; Transient; Plate

1. Introduction

The Isochoric, Counter-Current Heat Exchanger (ICHE) has been proposed by Georgiou [1,2] as a mechanism capable of implementing the thermodynamic process of regenerative preheating in the Lenoir cycle. As shown by Georgiou and Gkiouvetsis [3], the introduction of the mechanism may lead to a modified Otto cycle with a real efficiency comparable to that of the combined cycles employed in modern power plants.

The proposed heat exchanger consists of two parallel lid driven cavity cascades moving in opposite directions on the two sides of the separating plate, which acts as their common lid. The cavities of the first cascade are filled with the hot gas, while the cavities in the opposite cascade are filled with the cold one. Heat is transferred from the hot gas cavities to the cold ones through the plate. In each cavity a temperature change is associated with a corresponding pressure one, in agreement with the state law of the (nearly) perfect gasses. The length of the cascade depends on the temperature difference, the overall thermal resistance and the exposed area of the lid plate. Its length grows with the thermal resistance of the lid. In order to minimize resistance, the lid plate has to be thin and highly conductive. This, however, is limited by structural considerations. The very small thickness of this plate means that the heat flux through it essentially will be one-dimensional, although a small degree of a “travelling source” effects will be present. The complete heat conduction process through this plate is highly transient and complex. A number of mechanisms are involved. The gas temperatures exhibit step changes as the cavity ribs pass over a given point and ramp ones due to the continuous heating/cooling of the gas inside each cavity. The cavity wall heat transfer coefficients are nearly constant away from the ribs, but they change significantly near them. Hence, the optimisation of the design process requires a good under-

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Nomenclature

A	heat transfer surface	m^2
B	Biot number	
c	specific heat	$J{\cdot}kg^{-1}{\cdot}K^{-1}$
E	accumulated energy	J
h	heat transfer coefficient	$W{\cdot}m^{-2}{\cdot}K^{-1}$
h_{tot}	overall heat transfer coefficient	$W{\cdot}m^{-2}{\cdot}K^{-1}$
h_p	internal conductance of the plate per unit length	$W{\cdot}m^{-2}{\cdot}K^{-1}$
k	thermal conductivity	$W{\cdot}m^{-1}{\cdot}K^{-1}$
q	heat flux	$W{\cdot}m^{-2}$
Q	instantaneous energy accumulation rate per unit area	$W{\cdot}m^{-2}$
R	dimensionless recovery time	
T	temperature	K
$T_{\infty 1}$	temperature of fluid 1	K
$T_{\infty 2}$	temperature of fluid 2	K
t	time	s
t_d	delay period	s
w	plate thickness	m
y	transverse distance	m
Y	dimensionless y coordinate	
Y_0	coordinate of a point with final temperature perturbation equal to zero	

Greek symbols

α	thermal diffusivity	$m^2{\cdot}s^{-1}$
β	ratio of the overall heat transfer coefficient to the internal conductance per unit length of the plate, $= h_{tot}/h_p$	
γ	magnitude of the step change in the convective heat transfer coefficient	
$\mathcal{E}$	dimensionless accumulated energy	
ε	very small positive value involved in the definition of the recovery time	
θ	dimensionless temperature	
λ, ξ	eigenvalues involved in the analytical solution	
ρ	mass density	$kg{\cdot}m^{-3}$
τ	dimensionless time	
τ_d	dimensionless delay period between the step change in the convective heat transfer coefficients	
τ_d^*	dimensionless delay period satisfying a specific condition	
φ	dimensionless heat flux perturbation	
Φ	dimensionless instantaneous energy accumulation rate	

Subscripts

1	side of the plate in contact with the fluid 1, at $y = 0$ ($Y = 0$)
2	side of the plate in contact with the fluid 2, at $y = w$ ($Y = 1$)
0	initial
01	initial, at $y = 0$ ($Y = 0$)
02	initial, at $y = w$ ($Y = 1$)
f, F	final
in	entering the plate
out	exiting the plate
p	thin plate

standing of each of the above separate mechanisms as well as their mutual interaction.

The transient heat transfer through an infinite flat plate has been studied extensively in the past. Carlsaw and Jeager [4], Schneider [5] and Ozisik [6] provide extensive reviews on these studies. The reported cases, however, are limited to step changes on one side only and of course there is no mention of any "phase difference". The extension to the travelling source has also been studied, by assuming that the source travels above only one of the two sides of the plate and for various boundary conditions [4,7]. Georgiou and Siakavellas [8] have studied the case of step changes in the two free stream temperatures and the corresponding time delay effects.

The present study investigates the simplified limiting case of the transient, one-dimensional conduction through an infinite flat plate when the gasses in contact with each of its two sides are exhibiting a step change in their heat transfer coefficients. These changes correspond to the effect created by a rib passing over a given point of the lid plate. If the width of the rib is small, this change takes the form of a step. Of course, since the two cavity cascades travel in opposite directions the step changes do not occur concurrently, but with a delay period.

2. Definition and analysis of the problem

An infinite flat plate of thickness w separating two fluids is considered (Fig. 1a) as the idealization of the separating plate in the counter moving cavity cascade concept (Fig. 1b) of the Isochoric Counter-Current Heat Exchanger [1]. At a given x station of the plate, the passing of a cavity separating rib modifies the local free stream conditions and the corresponding heat transfer coefficients. The two fluids are at different temperatures. Actually it is $T_{\infty 2} > T_{\infty 1}$, so that heat is transferred from fluid 2 to fluid 1 through the plate. The plate material properties, i.e., the mass density (ρ), the specific heat (c), the thermal conductivity (k) and thermal diffusivity ($\alpha = k/\rho c$) are considered uniform throughout the plate, independent of temperature and constant with time. Initially ($t \leqslant 0$) the plate is considered to be in equilibrium with

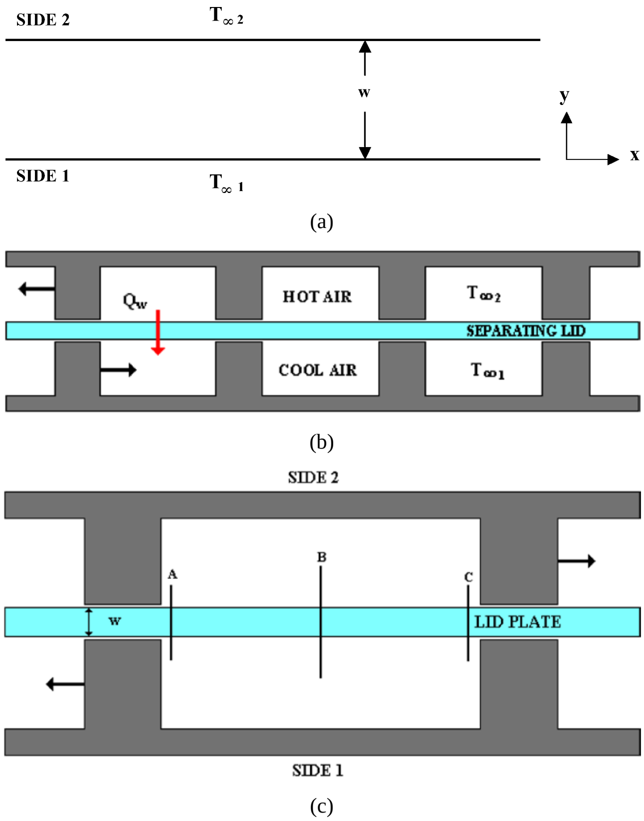


Fig. 1. Schematic diagram of the plate and the heat exchanger. (a) The infinite flat plate and the surrounding fluids; (b) the isochoric, counter-current heat exchanger concept; (c) the relative positions of the points on the separating wall.

its environment, i.e., steady state temperature and heat flux has been established. At $t > 0$, the convective heat transfer coefficients, initially $h_1 = h_{01}$ and $h_2 = h_{02}$, undergo step changes with a delay period t_d among them, leading to new values $h_1 = \gamma_1 h_{01}$ and $h_2 = \gamma_2 h_{02}$, respectively. The nature of this time delay is illustrated in Fig. 1c. The point A will sense the passing of the rib separating the cavities in the hot gas cascade first and the rib on the opposite cascade will follow. In the case of the point C the sequence of the events will be inverted. In the center point B the passing of the two ribs will be felt simultaneously.

The heat transfer inside the finite thickness plate obeys Fick's equation

$$k\frac{\partial^2 T}{\partial y^2} = \rho c\frac{\partial T}{\partial t} \tag{1}$$

where the direction y is normal to the two surfaces of the plate and $y = 0$ corresponds to the surface exposed to the fluid 1 (Fig. 1a). The boundary and the initial conditions are:

$$k\frac{\partial T}{\partial y} = \gamma_1 h_{01}(T - T_{\infty 1}), \quad y = 0 \tag{2a}$$

$$k\frac{\partial T}{\partial y} = \gamma_2 h_{02}(T_{\infty 2} - T), \quad y = w \tag{2b}$$

$$T(y,t) = T_0(y), \quad t = 0 \tag{3}$$

The initial temperature distribution, $T_0(y)$, is given by Eq. (A.8) in Appendix A.

It is assumed that the first step change appears at $t = 0^+$ and the second at $t = t_d^+$. When h_1 changes first, $\gamma_1 = 1$ for $t \leqslant 0$ and $\gamma_1 > 1$ for $t > 0$, while $\gamma_2 = 1$ for $t \leqslant t_d$ and $\gamma_2 > 1$ for $t > t_d$. When h_2 changes first, $\gamma_1 = 1$ for $t \leqslant t_d$ and $\gamma_1 > 1$ for $t > t_d$, while $\gamma_2 = 1$ for $t \leqslant 0$ and $\gamma_2 > 1$ for $t > 0$. Of course, if the two step changes appear simultaneously ($t_d = 0$), then $\gamma_1 = \gamma_2 = 1$ for $t \leqslant 0$ and both $\gamma_1, \gamma_2 > 1$ for $t > 0$.

The local temperature perturbation, $T(y,t) - T_0(y)$ may be transformed into the non-dimensional temperature $\theta(Y,\tau)$:

$$\theta(Y,\tau) = \frac{T(y,t) - T_0(y)}{T_{\infty 2} - T_{\infty 1}} \tag{4}$$

which together with the non-dimensional parameters of distance (Y) and time (τ)

$$Y \equiv \frac{y}{w}, \quad \tau \equiv \frac{\alpha t}{w^2} \tag{5}$$

modify Eqs. (1)–(3) into the following non-dimensional form (Appendix A):

$$\frac{\partial^2\theta}{\partial Y^2} = \frac{\partial\theta}{\partial\tau}, \quad 0 < Y < 1 \tag{6}$$

$$\begin{aligned}\frac{\partial\theta}{\partial Y} &= \gamma_1 B_{01}\theta + (\gamma_1 - 1)\frac{B_0}{1+B_0}\\ &= \gamma_1 B_{01}\theta + (\gamma_1 - 1)\beta_0, \quad Y = 0\end{aligned} \tag{7a}$$

$$\begin{aligned}\frac{\partial\theta}{\partial Y} &= -\gamma_2 B_{02}\theta + (\gamma_2 - 1)\frac{B_0}{1+B_0}\\ &= -\gamma_2 B_{02}\theta + (\gamma_2 - 1)\beta_0, \quad Y = 1\end{aligned} \tag{7b}$$

$$\theta(Y,\tau) = 0, \quad \tau = 0 \tag{8}$$

The initial Biot numbers B_{01}, B_{02} and B_0 involved in Eqs. (7a) and (7b) are defined as

$$B_{01} = \frac{h_{01}}{h_p} = \frac{h_{01}w}{k}, \qquad B_{02} = \frac{h_{02}}{h_p} = \frac{h_{02}w}{k}$$
$$B_0 = \frac{h_0}{h_p} = \frac{h_0 w}{k} \tag{9}$$

where $h_p = k/w$ is the internal conductance per unit length of the plate, while h_0 is the initial "combined convective" heat transfer coefficient. Actually, h_0 is a part of the initial "overall heat transfer coefficient" ($h_{0,tot}$), given as:

$$\frac{1}{h_0} = \frac{1}{h_{01}} + \frac{1}{h_{02}}$$
$$\frac{1}{h_{0,tot}} = \frac{1}{h_0} + \frac{1}{h_p} = \frac{1}{h_{01}} + \frac{1}{h_{02}} + \frac{1}{h_p} \tag{10}$$

Consequently, the effective Biot number B_0 and the ratio β_0, involved in Eqs. (7a), (7b), are:

$$B_0 = \frac{B_{01}B_{02}}{B_{01} + B_{02}} = \frac{h_0}{h_p}$$
$$\beta_0 = \frac{B_0}{1+B_0} = \frac{h_0}{h_0 + h_p} = \frac{h_{0,tot}}{h_p} \tag{11}$$

The final steady state temperature, $\theta_f(Y) \equiv \theta(Y, \tau \to \infty)$ is the solution of

$$\left.\frac{\partial \theta_f}{\partial Y}\right|_{Y=0} = \left.\frac{\partial \theta_f}{\partial Y}\right|_{Y=1} = \theta_f(1) - \theta_f(0) \tag{12}$$

Replacing for the boundary conditions (Eqs. (7a), (7b)), Eq. (12) yields

$$\theta_f(Y) = (\beta - \beta_0)\left[Y + \frac{(\gamma_2 - \gamma_1) - (\gamma_1 - 1)B_2}{(\gamma_2 - 1)B_1 + (\gamma_1 - 1)B_2}\right] \tag{13}$$

The Biot numbers, i.e., $B_1 = \gamma_1 B_{01}$, $B_2 = \gamma_2 B_{02}$, B and the ratio β are defined similarly to B_{01}, B_{02}, B_0 and β_0 by

$$B_1 = \frac{h_1}{h_p} = \frac{h_1 w}{k}, \qquad B_2 = \frac{h_2}{h_p} = \frac{h_2 w}{k}$$
$$B = \frac{B_1 B_2}{B_1 + B_2} = \frac{h}{h_p} = \frac{hw}{k}$$
$$\beta = \frac{B}{1+B} = \frac{h}{h + h_p} = \frac{h_{tot}}{h_p} \tag{14}$$

The combined convective (h) and overall (h_{tot}) heat transfer coefficients are given by

$$\frac{1}{h} = \frac{1}{h_1} + \frac{1}{h_2}$$
$$\frac{1}{h_{tot}} = \frac{1}{h} + \frac{1}{h_p} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_p} \tag{15}$$

As it will be clear below (Eq. (23)), β_0 and β are the non-dimensional initial and final steady state heat fluxes for the thick plate, respectively, while B_0 and B the corresponding ones for the thin plate limit.

In general, there exists a point $Y = Y_0$ where the final temperature perturbation is zero: $\Delta\theta(Y_0) = \theta_f(Y_0) - \theta_0(Y_0) = \theta_f(Y_0) = 0$. This is obtained from Eq. (13) when $\theta_f(Y_0) = 0$, i.e.,

$$Y_0 = \frac{\gamma_1 - \gamma_2 + (\gamma_1 - 1)B_2}{(\gamma_2 - 1)B_1 + (\gamma_1 - 1)B_2} \tag{16}$$

Eq. (13) is now written in terms of Y_0 as:

$$\theta_f(Y) = (\beta - \beta_0)(Y - Y_0) \tag{17}$$

The conditions for which the point Y_0 lies within the plate (i.e., $0 \leqslant Y_0 \leqslant 1$) depend upon the modified Biot numbers, i.e., on γ_1 and γ_2. The two extreme cases, $Y_0 = 0$ or $Y_0 = 1$, occur when:

$$\gamma_1 = \frac{\gamma_2 + B_2}{1 + B_2} \quad (Y_0 = 0)$$
$$\gamma_2 = \frac{\gamma_1 + B_1}{1 + B_1} \quad (Y_0 = 1) \tag{18}$$

The position Y_0 separates the plate into two regions with higher/lower thermal stresses at the $\tau \to \infty$ limit, as compared to the initial thermal stress distribution.

The heat flux, $q(y,t) = -k\partial T(y,t)/\partial y$, is transformed into the non-dimensional heat flux $\varphi(Y,\tau) = q(y,t)/q_m$, where

$$q_m = -h_p(T_{\infty 2} - T_{\infty 1}) = -k\frac{T_{\infty 2} - T_{\infty 1}}{w} \tag{19}$$

By taking into consideration Eqs. (A.6) and (A.9) (see Appendix A), $\varphi(Y,\tau)$ is written as:

$$\varphi(Y,\tau) = \frac{q(y,t)}{q_m} = \frac{\partial\theta(Y,\tau)}{\partial Y} + \beta_0 \tag{20}$$

The non-dimensional form of the surface heat fluxes into the plate (at $y = w$), $q_{in}(t) = q(w,t)$, and out of the plate (at $y = 0$), $q_{out}(t) = q(0,t)$, is obtained from Eq. (20), for $Y = 1$ and $Y = 0$ as:

$$\varphi_{in}(\tau) \equiv \varphi(1,\tau) = \frac{q_{in}(t)}{q_m} = \left.\frac{\partial\theta(Y,\tau)}{\partial Y}\right|_{Y=1} + \beta_0$$
$$= \gamma_2\left[\beta_0 - B_{02}\theta(1,\tau)\right] \tag{21}$$

$$\varphi_{out}(\tau) \equiv \varphi(0,\tau) = \frac{q_{out}(t)}{q_m} = \left.\frac{\partial\theta(Y,\tau)}{\partial Y}\right|_{Y=0} + \beta_0$$
$$= \gamma_1\left[\beta_0 + B_{01}\theta(0,\tau)\right] \tag{22}$$

where Eqs. (7b) and (7a) have been used for $\partial\theta/\partial Y$ at $Y = 1$ and $Y = 0$, respectively. If we take into consideration that for $\tau = 0$ we have $\partial\theta/\partial Y = 0$ (Eqs. (7a), (7b) and (8)), while for $\tau \to \infty$ we have $\partial\theta_f/\partial Y = \beta - \beta_0$ (Eq. (17)), the initial (φ_0) and final (φ_f) steady state heat fluxes become:

$$\varphi_0 = \varphi_{in}(0) = \varphi_{out}(0) = \beta_0$$
$$\varphi_f = \varphi_{in}(\infty) = \varphi_{out}(\infty) = \beta \tag{23}$$

where β_0 and β are given by Eqs. (11) and (14), respectively. In the thin plate limit (Section 3), we have $B_0 \ll 1$ and $B \ll 1$ since $h_p \gg h$. Hence, $\varphi_0 = \beta_0 \approx B_0$ and $\varphi_f = \beta \approx B$.

The temperature perturbation (θ) leads to an energy accumulation inside the plate. The instantaneous energy accumulation rate per unit area (Q) is equal to the difference between the surface heat fluxes, i.e., $Q(t) = q_{in}(t) - q_{out}(t)$. In non-dimensional terms this is transformed, according to Eqs. (21) and (22), into

$$\Phi(\tau) = \frac{Q(t)}{q_m} = \varphi_{in}(\tau) - \varphi_{out}(\tau)$$
$$= (\gamma_2 - \gamma_1)\beta_0 - \gamma_2 B_{02}\theta(1,\tau) - \gamma_1 B_{01}\theta(0,\tau) \tag{24}$$

If the plate is divided into elementary slabs, of thickness dy, volume $dV = A\,dy$ and mass $dm = \rho A\,dy$ (A is the exposed surface area), the elementary amount of energy that has been accumulated inside the slab at position y over the period $(0,t)$ is

$$dE(y,t) = dm\,c\Delta T(y,t) = \rho Ac\big[T(y,t) - T_0(y)\big]dy \tag{25}$$

The total amount of energy that has been accumulated inside the plate over the period $(0,t)$ and within a volume $V = Aw$ with mass $m = \rho Aw$, is obtained by integrating Eq. (25):

$$E(t) = \int_V dm\,c\Delta T = \rho Ac\int_{y=0}^{y=w}\big[T(y,t) - T_0(y)\big]dy \tag{26}$$

In non-dimensional terms, this parameter is transformed into $\mathcal{E}(\tau)$, where

$$\mathcal{E}(\tau) = \frac{E(t)}{mc(T_{\infty 2} - T_{\infty 1})} = \int_{Y=0}^{Y=1} \theta(Y,\tau)\, dY \tag{27}$$

In the limit $\tau \to \infty$ and in conjunction with Eq. (17), Eq. (27) yields

$$\mathcal{E}_f = \mathcal{E}(\infty) = \int_{Y=0}^{Y=1} \theta_f(Y)\, dY = (\beta - \beta_0)\left(\frac{1}{2} - Y_0\right) \tag{28}$$

By comparing Eqs. (28) and (17) we observe that

$$\mathcal{E}_f = \theta_f(0.5) \tag{29}$$

Consequently, if $Y_0 = 0.5$, the final energy accumulation is zero: $\mathcal{E}_f = 0$.

3. The thin plate limit

When $h_p \gg h$, the temperature gradients within the plate may be neglected. In such a case the plate is labelled as "thermally thin" and the instantaneous plate temperature is almost uniform, i.e., $T(y,t) \approx T_p(t)$. The instantaneous energy balance over a given control volume of unity area and thickness w yields the governing equation for the thin plate limit:

$$\begin{aligned}\rho c w \frac{dT_p}{dt} &= q_{in}(t) - q_{out}(t) \\ &= \gamma_2 h_{02}(T_{\infty 2} - T_p) - \gamma_1 h_{01}(T_p - T_{\infty 1})\end{aligned} \tag{30}$$

The non-dimensional form of Eq. (30) in terms of the perturbation (θ_p) becomes (see Appendix A):

$$\frac{d\theta_p}{d\tau} + (\gamma_1 B_{01} + \gamma_2 B_{02})\,\theta_p(\tau) = (\gamma_2 - \gamma_1)\, B_0 \tag{31}$$

with initial condition $\theta_p(\tau) = 0$ at $\tau = 0$. The solution of Eq. (31) is:

(i) For $0 \leqslant \tau \leqslant \tau_d$:

- if h_1 changes first, i.e., $h_1 = \gamma_1 h_{01}$, $h_2 = h_{02}$

$$\theta_p(\tau) = \frac{B_0}{B_1 + B_{02}}(1 - \gamma_1)\left[1 - \mathrm{e}^{-(B_1 + B_{02})\tau}\right] \tag{32a}$$

- if h_2 changes first, i.e., $h_1 = h_{01}$, $h_2 = \gamma_2 h_{02}$

$$\theta_p(\tau) = \frac{B_0}{B_{01} + B_2}(\gamma_2 - 1)\left[1 - \mathrm{e}^{-(B_{01} + B_2)\tau}\right] \tag{32b}$$

(ii) For $\tau > \tau_d$ ($h_1 = \gamma_1 h_{01}$, $h_2 = \gamma_2 h_{02}$):

$$\theta_p(\tau) = \theta_F + \left[\theta_p(\tau_d) - \theta_F\right]\mathrm{e}^{-(B_1 + B_2)(\tau - \tau_d)} \tag{33}$$

where $\theta_p(\tau_d)$ is the temperature perturbation at $\tau = \tau_d$, given by either Eq. (32a) or (32b), and θ_F is the final steady-state value, given by

$$\theta_F = \frac{\gamma_2 - \gamma_1}{\gamma_1 B_{01} + \gamma_2 B_{02}} B_0 \tag{34}$$

The recovery time is defined as the time taken after $\tau = 0$ to reach the final steady state temperature, i.e., $\theta_p(\tau) = \theta_F$ or $d\theta_p/d\tau = 0$. The time differentiation of Eq. (33) gives:

$$\begin{aligned}\frac{d\theta_p}{d\tau} &= \left[B_0(\gamma_2 - \gamma_1) - \theta_p(\tau_d)(B_1 + B_2)\right]\mathrm{e}^{-(B_1 + B_2)(\tau - \tau_d)} \\ &= 0\end{aligned} \tag{35}$$

Strictly speaking, this is accomplished only in the limit $\tau \to \infty$, since the plate temperature tends asymptotically to its final steady state value. However, for practical purposes, it may be assumed that thermal equilibrium is established when the time derivative of the plate temperature is smaller than or at least equal to a small critical value ε:

$$\left|\frac{d\theta_p(\tau)}{d\tau}\right| \leqslant \left(\frac{\Delta\theta_p}{\Delta\tau}\right)_{\min} \equiv \varepsilon \tag{36}$$

The recovery time, R_p, is defined as the elapsed period after which the condition (36) is continuously satisfied. Eqs. (35) and (36) give:

$$\begin{aligned}R_p = \tau_d &+ \frac{1}{B_1 + B_2} \\ &\times \ln\left|\frac{B_0(\gamma_2 - \gamma_1) - \theta_p(\tau_d)(B_1 + B_2)}{\varepsilon}\right|\end{aligned} \tag{37}$$

It is possible to have $d\theta_p/d\tau = 0$, if the expression $[B_0(\gamma_2 - \gamma_1) - \theta_p(\tau_d)(B_1 + B_2)]$ in Eq. (35) is equal to zero. This unique delay period (τ_d^*) that forces a steady state condition is given, if h_1 changes first, by

$$\tau_d^* = \frac{1}{B_1 + B_{02}} \ln\left[\frac{\gamma_1 - 1}{\gamma_1(\gamma_2 - 1)} \frac{B_1 + B_2}{B_{01} + B_{02}}\right] \tag{38a}$$

or (if h_2 changes first) by

$$\tau_d^* = \frac{1}{B_{01} + B_2} \ln\left[\frac{\gamma_2 - 1}{\gamma_2(\gamma_1 - 1)} \frac{B_1 + B_2}{B_{01} + B_{02}}\right] \tag{38b}$$

The estimation of the heat transfer coefficients, h_1 and h_2, was based on the expression $h = \rho_f \cdot c_p \cdot u \cdot St$, where ρ_f is the mass density of the fluids 1 or 2 (cold or hot gas, respectively), with an expected range from $\rho_f = 1.2$ to 12 $\mathrm{kg/m^3}$ in the application considered (i.e., the ICHE). The corresponding heat capacities are between $c_p = 1000$ and 1200 J/(kg·K), while the Stanton number varies from $St = 0.002$ to 0.006 (when the surface of the separating plate is made sufficiently "rough"). The cavity cascades of the isochoric heat exchanger move with a velocity going from $u =$ 20 to 50 m/s. The combination of the above data yield a minimum value for the heat transfer coefficients 48 $\mathrm{W/(m^2{\cdot}K)}$ and a maximum one 4320 $\mathrm{W/(m^2{\cdot}K)}$. Since the thickness of the separating plate is expected to vary from $w = 1$ to 5 mm, the expected minimum and maximum Biot numbers are 0.004 and 1.7, respectively, provided that the plate material is chrome–nickel steel. The maximum Biot number may be extended up to 3, if a titanium alloy (Ti-6Al-4V) is used instead of chrome–nickel. The Biot numbers considered in the calculations are of the order of 0.1 for the thin plate approximation, and 1.0 in the thick plate analysis (Section 4).

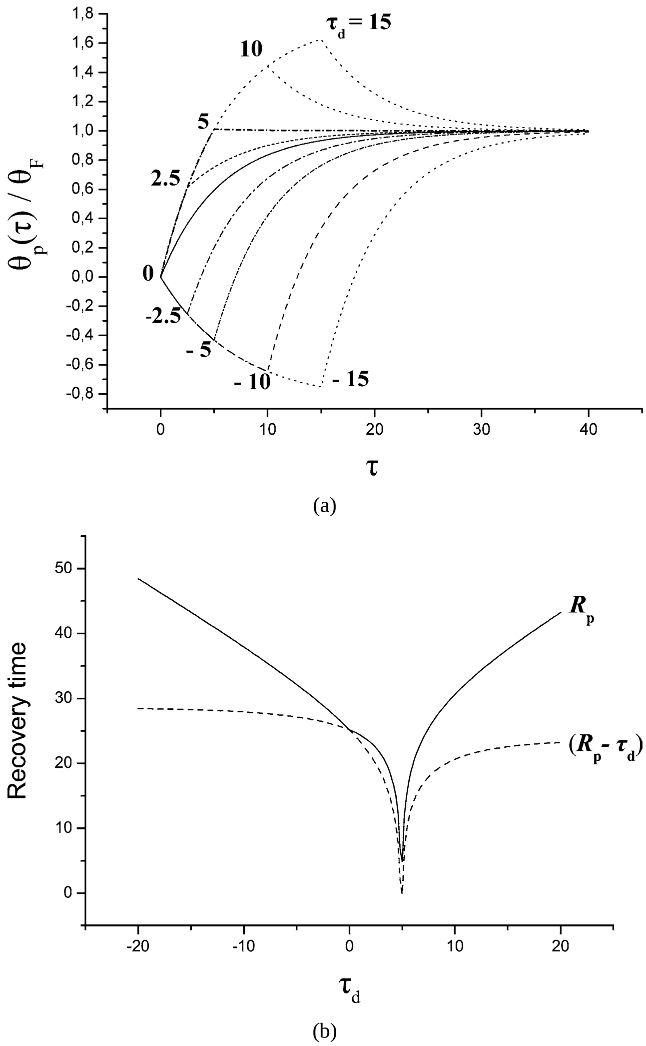


Fig. 2. The influence of the delay period (τ_d) on: (a) the evolution of the temperature perturbation, (b) the recovery time for the thin plate.

Hence, in the thin plate limit, the following values are considered: $B_{01} = 0.05$, $B_{02} = 0.08$, $\gamma_1 = 1.2$ and $\gamma_2 = 1.5$ (which means $B_1 = 0.06$ and $B_2 = 0.12$). The asymmetry of the above values is due to the different densities in the two sides. The side 1 gas has undergone an isentropic expansion before entering the corresponding side cavities in the isochoric heat exchanger.

The influence of the delay period on the evolution of the ratio θ_p/θ_F is illustrated in Fig. 2a. The negative τ_d values are employed in the figure to indicate that the step change appears initially in the h_1 coefficient. When the step change appears initially in the h_2 one, positive values are assigned to τ_d. In the actual calculations τ_d is always positive. For $t \to \infty$ the ratio θ_p/θ_F tends towards unity (Fig. 2a). When, however, the second step appears the perturbation is quite significant. The data show clearly that, during the transition period, the perturbation deviates significantly from the steady state values. The perturbation actually may either overshoot temporarily the final steady state value or undershoot the initial (zero) value by a wide margin. In a periodic appearance of these steps, the deviations not only will affect the heat resistance but they will introduce significant thermal stresses in the plate material as well. As predicted by Eqs. (38a), (38b) for the above selected parameter values, for $\tau_d \approx 5$ (actually $\tau_d^* = 4.92$) and as soon as the second step appears, the thin plate temperature remains constant (Fig. 2a), since $d\theta_p/d\tau = 0$ for $\tau > \tau_d$. This is corroborated by the recovery time, R_p (Fig. 2b), which exhibits a singular performance near that point, as predicted by Eq. (37). The influence of the delay period to the recovery time is understood better if we study the time elapsed from the second step change to the thermal equilibrium, i.e., the time difference, $(R_p - \tau_d)$, as a function of τ_d (dashed line). We observe that $(R_p - \tau_d)$ away from the singular point increases very slowly with τ_d.

4. The thermally thick plate

In the case of the thermally thick plate one needs to solve a non-homogeneous problem (Eqs. (6)–(8)), with boundary conditions of the third kind as discussed by Ozisik [6]. Details of the solution process are given in Appendix B. The analytic solution $\theta(Y, \tau)$, is:

(a) Over the time interval $0 \leqslant \tau \leqslant \tau_d$:

$$\theta(Y, \tau) = 2\beta_0 \sum_{m=1}^{\infty} f(\xi_m) \left(\cos \xi_m Y + \frac{\gamma_1 B_{01}}{\xi_m} \sin \xi_m Y \right) \times \left(1 - e^{-\xi_m^2 \tau}\right) \tag{39}$$

The eigenvalues ξ_m are the positive roots of one of the following transcendental equations:

$$\tan \xi = \frac{\xi (B_1 + B_{02})}{\xi^2 - B_1 B_{02}} \tag{40a}$$

or

$$\tan \xi = \frac{\xi (B_{01} + B_2)}{\xi^2 - B_{01} B_2} \tag{40b}$$

depending on whether h_1 (Eq. (40a)) or h_2 (Eq. (40b)) changes first. For the latter case, in Eq. (39) it is set $\gamma_1 = 1$. The functions $f(\xi_m)$ are given in Appendix B.

(b) For $\tau > \tau_d$:

$$\begin{aligned} \theta(Y, \tau) = {} & 2\beta_0 \sum_{n=1}^{\infty} C(\lambda_n) \\ & \times \left(\cos \lambda_n Y + \frac{B_1}{\lambda_n} \sin \lambda_n Y \right) e^{-\lambda_n^2 (\tau - \tau_d)} \\ & \times \sum_{m=1}^{\infty} f(\xi_m) D(\lambda_n, \xi_m) \left(1 - e^{-\xi_m^2 \tau_d}\right) \\ & + 2\beta_0 \sum_{n=1}^{\infty} g(\lambda_n) \left(\cos \lambda_n Y + \frac{B_1}{\lambda_n} \sin \lambda_n Y \right) \\ & \times \left[1 - e^{-\lambda_n^2 (\tau - \tau_d)}\right] \end{aligned} \tag{41}$$

The eigenvalues λ_n are the positive roots of the transcendental equation

$$\tan\lambda = \frac{\lambda(B_1 + B_2)}{\lambda^2 - B_1 B_2} \tag{42}$$

while the functions $f(\xi_m)$, $g(\lambda_n)$, $C(\lambda_n)$ and $D(\lambda_n, \xi_m)$ are given in Appendix B. If the step changes to h_1 and h_2 appear simultaneously, then $\tau_d = 0$ and the solution becomes:

$$\theta(Y,\tau) = 2\beta_0 \sum_{n=1}^{\infty} g(\lambda_n)\left(\cos\lambda_n Y + \frac{B_1}{\lambda_n}\sin\lambda_n Y\right) \times \left(1 - \mathrm{e}^{-\lambda_n^2\tau}\right) \tag{43}$$

The non-dimensional heat flux $\varphi(Y,\tau)$ is given by Eq. (20). The series converge rapidly for the Biot numbers employed in the applications. In general, less than fifty terms are sufficient for the convergence. The present study employed eighty terms, the computation time being very short.

Figs. 3, 4 illustrate the transient evolution of the thick plate temperature perturbation and the influence of the delay period on it. The initial Biot numbers are taken equal to $B_{01} = 0.5$ and $B_{02} = 1.0$, i.e., $B_{02} = 2B_{01}$. These are typical values for the expected heat exchanger application. In Fig. 3, $\gamma_1 = 1.2$, $\gamma_2 = 1.5$. For this combination $Y_0 = 0$, i.e., the initial and final steady state temperature perturbations are zero at the bottom side of the plate. The evolution of the perturbation is given for three Y positions, i.e., $Y = 0.0, 0.5$ and 1.0. The deviation of the instantaneous perturbation value from the steady state values is comparable to the thin plate results for all Y positions. Both over and undershooting are observed, depending on the time delay period and the Biot numbers. The magnitude of these deviations is comparable to the steady state perturbation values. The phenomenon of sudden adjustment to a steady state similar to the predictions of Eqs. (38a), (38b) for the thin plate limit is observed here as well (for $\tau_d = 0.35$).

The combined influence of the time delay (τ_d) and the Biot number step change are illustrated in Fig. 4. The combined effect of the γ's is presented through the Y_0 magnitude. The following three combinations have been employed in this figure:

(i) $\gamma_1 = 1.2$ and $\gamma_2 = 1.5$ (i.e., $Y_0 = 0.0$);
(ii) $\gamma_1 = 1.2$ and $\gamma_2 = 1.25$ (i.e., $Y_0 = 0.5$);
(iii) $\gamma_1 = 1.2$ and $\gamma_2 = 1.125$ (i.e., $Y_0 = 1.0$).

The time-wise evolution of the temperature perturbation is tracked in three Y positions ($Y = 0.0,\ 0.5$ and 1.0), while only two delay periods are considered (a) $\tau_d = +1.0$, i.e., the step appears initially in the upper surface (solid line) and (b) $\tau_d = -1.0$, i.e., the first step occurs in the bottom surface (dashed line). The data indicate that the perturbation evolution is similar in all plate positions for a given (Y_0, τ_d) combination. The magnitudes, however, may differ by a respectable percentage. This implies that a thermal stress will always be present inside the plate material and for the whole duration of the transient part of the process. For a given set of (γ_1, τ_d) and when γ_2 increases the point Y_0 moves from

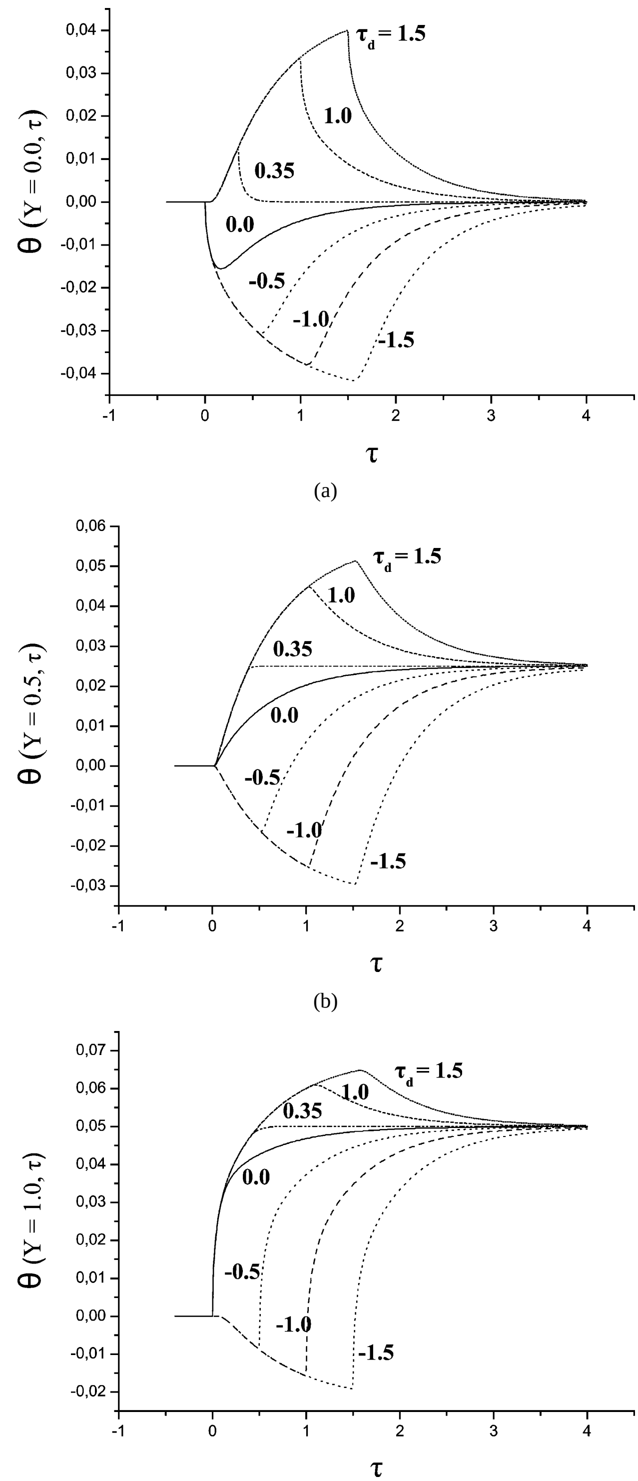


Fig. 3. The evolution of the temperature perturbation at the exposed surfaces and the middle of the plate for a thermally thick plate, when $Y_0 = 0.0$.

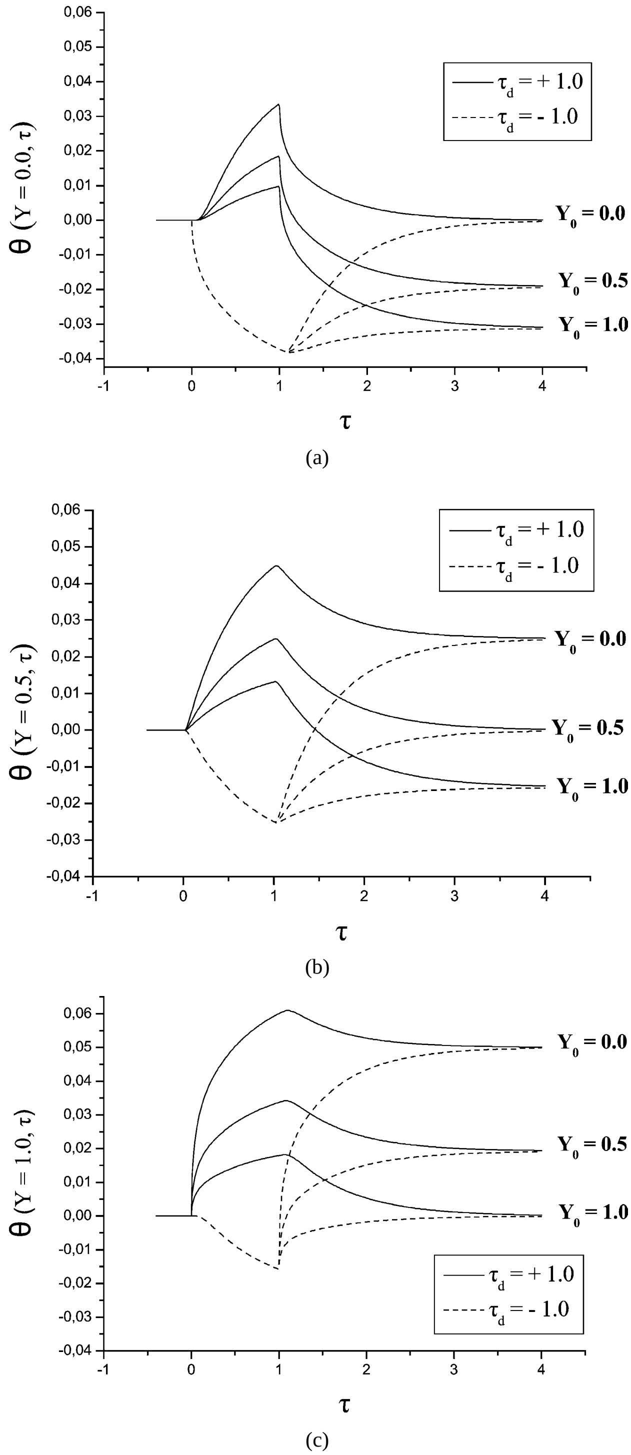
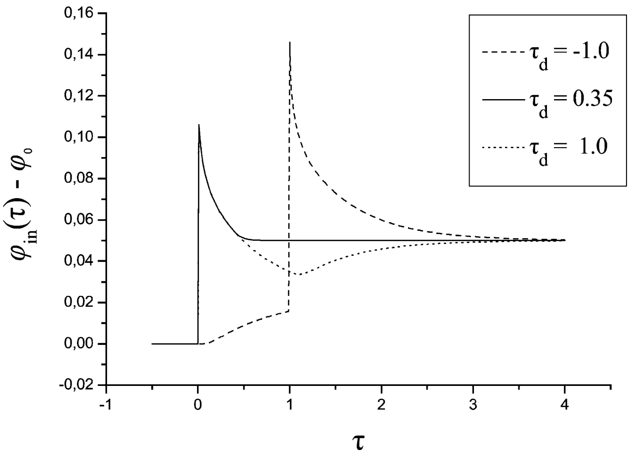


Fig. 4. The influence of the delay period (τ_d) and the Biot number step change on the temperature evolution.

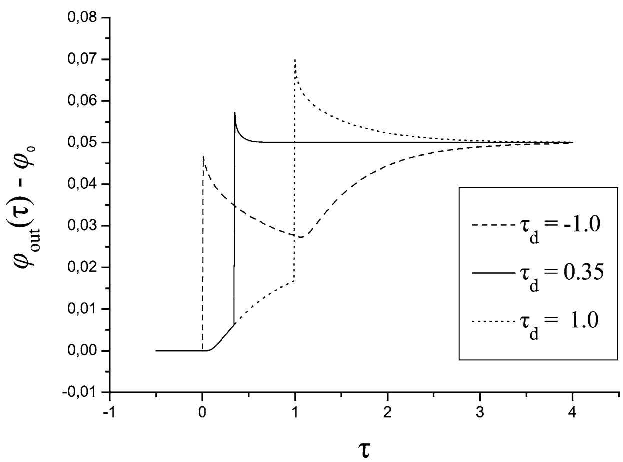
the top towards the bottom of the plate and the temperature increases. Of course, the situation is inverted when γ_1 increases, for a given set of (γ_2, τ_d). On the other hand, for a given set of (γ_1, γ_2), there exists a specific value τ_d^* for the delay period that forces to a steady state condition as soon as the second step appears. For γ_1 constant, this specific value increases with γ_2 if h_2 changes first, and decreases with γ_2 if h_1 changes first. To clarify these, let us apply the case (ii) ($\gamma_1 = 1.2$, $\gamma_2 = 1.25$), to a typical isochoric heat exchanger configuration. Since $B_{01} = 0.5$ and $B_{02} = 1.0$, we have $B_1 = 0.6$ and $B_2 = 1.25$. For the data given in Section 3, this configuration could correspond to a titanium alloy plate, with a thickness of $w = 2$ mm and a cavity velocity of $u = 45$ m/s. Then, the point for which the initial and final steady state temperature perturbations are zero is situated at $y_0 = 1.0$ mm (i.e., $Y_0 = 0.5$ in dimensionless terms). Since a typical distance between the cavities in the isochoric heat exchanger is 1.5 m, the delay period between the two steps is $t_d = 0.0167$ s. For this configuration the specific value of the parameter t_d^* is zero. If we consider for γ_2 a value slightly higher, i.e. $\gamma_2 = 1.26$, the calculations for this new combination ($\gamma_1 = 1.2$, $\gamma_2 = 1.26$), show the following: The point y_0 moves downwards, from $y_0 = 1.0$ mm to $y_0 = 0.94$ mm (i.e., $Y_0 = 0.47$ instead of 0.5), while the specific value of the parameter t_d^* is now 0.0167 s, i.e., equal to the delay period t_d, provided that h_2 changes first.

The dimensionless heat flux perturbations on the two exposed surfaces, i.e., $(\varphi_{in}(\tau) - \varphi_0)$ and $(\varphi_{out}(\tau) - \varphi_0)$, and the non-dimensional instantaneous energy accumulation rate, $\Phi(\tau) = \varphi_{in}(\tau) - \varphi_{out}(\tau)$, are illustrated in Fig. 5, which presents the heat flux evolution corresponding to Fig. 3 data. According to Eq. (23), the initial and final heat flux perturbations are 0 and $\beta - \beta_0$, respectively. Fig. 5 shows a sudden change in all instantaneous heat flux parameters at the moment each step is applied, followed by an exponential decay to the steady state values. If the delay period is relatively short, the first step flux perturbation has not die out and this leads to a significant shifting of the second step maxima. In turn, this implies that in a periodic application of such steps (as expected in the isochoric heat exchanger), the phenomena could be exploited to modify the entire "steady state" heat flux by a considerable percentage of the order of 10 to 20%. On the other hand, the phenomenon of sudden adjustment to a steady state at the introduction of the second step (corresponding to a $\tau_d = 0.35$ for the present configuration) may decrease sharply the "transient" period.

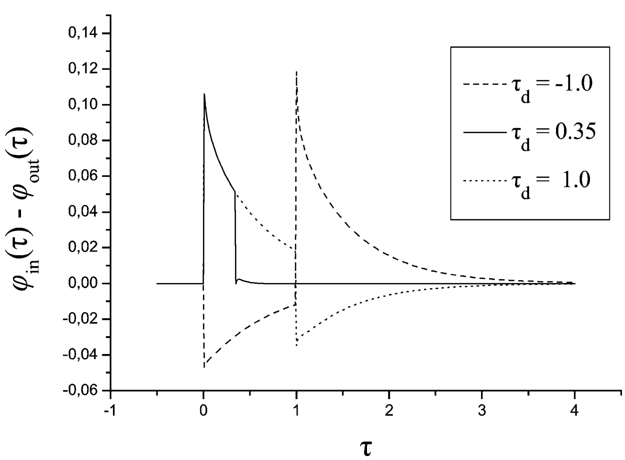
The accumulated energy, $\mathcal{E}(\tau)$, during the transient period is illustrated in Fig. 6a for the same conditions presented in Figs. 3 and 5. It is quite apparent that the amount of this energy may reach significant levels, up to 10% of the steady state "accumulation" for the present configuration. For different Y_0 values it may reach higher magnitudes. The evolution of this parameter is very similar to the thin plate temperature perturbation one (Fig. 2a), as well as to the thick plate temperature perturbation when $Y = 0.5$ (Fig. 3; see Eq. (29)). We verify again that if τ_d takes the specific value 0.35, the stored energy reaches its final value $\mathcal{E}_f$ almost immediately after the second step. The combined influence of the (Y_0, τ_d) parameters is exhibited in Fig. 6b, where the selected parameters are the same with those of Fig. 4. The influence is similar to that on the temperature perturbation.



(a)

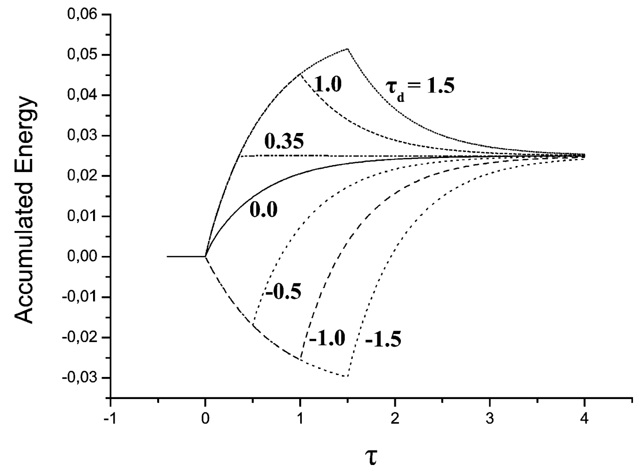


(b)

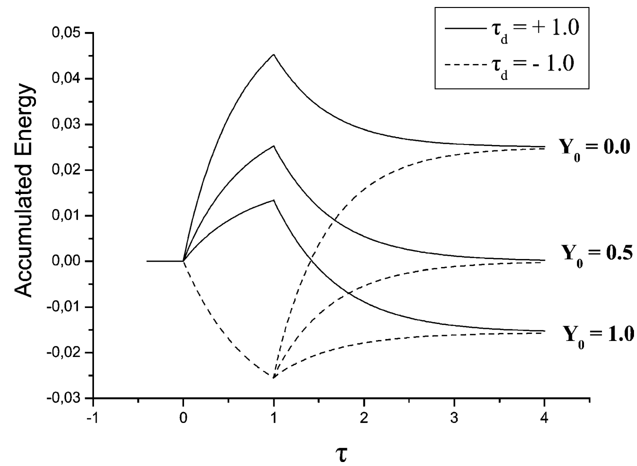


(c)

Fig. 5. The evolution of the heat flux perturbation on the exposed surfaces (a) $\varphi_{in}(\tau) - \varphi_0$, (b) $\varphi_{out}(\tau) - \varphi_0$ and (c) the instantaneous heat accumulation, when $Y_0 = 0.0$.



(a)



(b)

Fig. 6. (a) The influence of the delay period on the evolution of the accumulated energy inside the plate when $Y_0 = 0.0$. (b) The combined influence of Y_0 and the delay period on the accumulated energy.

According to Eq. (28), the energy that finally is stored to the plate is positive if $Y_0 < 0.5$, negative if $Y_0 > 0.5$ and zero if $Y_0 = 0.5$. So, for the case $Y_0 = 0$ it takes the value $\mathcal{E}_f = (\beta - \beta_0)/2$, while for $Y_0 = 1$ we have $\mathcal{E}_f = -(\beta - \beta_0)/2$. Of course, if $Y_0 < 0$, then $\mathcal{E}_f > (\beta - \beta_0)/2$, while if $Y_0 > 1$ we have $\mathcal{E}_f < -(\beta - \beta_0)/2$. These are evident from Fig. 6b, where the curves for the time evolution of the dimensionless energy stored for the three cases are shown in the same diagram (with $\tau_d = -1$ and $\tau_d = 1$). It is clear that $\mathcal{E}_f$ decreases as the point Y_0 moves from the bottom towards the top of the plate.

As it has already been mentioned, each combination of the Biot number step changes, γ_1 and γ_2, gives a different Y_0 point (Eq. (16)). If this lies within the plate, there exists a specific value τ_d^* for the delay period that drives the temperature, the instantaneous energy accumulation rate and the energy stored to take their final steady state values as soon as the second step is applied. This specific value is $\tau_d^* = 0.35$ when $Y_0 = 0$ (Figs. 3, 5, 6a), while for $Y_0 = 0.5$ and $Y_0 = 1$ the corresponding values are $\tau_d^* = 0.0$ and $\tau_d^* = -0.45$, respectively. The influence of τ_d on the above parameters is understood better if we examine the dependence of the thick plate recovery time $R(Y)$ on the delay period. $R(Y)$ is defined by a condition similar to Eq. (36), as the elapsed time after which the condition $|\partial\theta(Y,\tau)/\partial\tau| \leqslant \varepsilon$ is continuously satisfied.

Fig. 7 illustrates the influence of the delay period and Y_0 on the recovery time (actually the difference $R(Y) - \tau_d$), for $\varepsilon = 0.001$. The finite thickness of the plate does not distort the thin plate results to any large extent, since the recovery time is nearly the same over the greater part of the plate width. The presence of a singular point is quite apparent. This point corresponds to the specific values of the delay period $\tau_d^* = 0.35$, 0.0 and -0.45 when $Y_0 = 0.0$, 0.5 and 1.0, respectively, for which the temperature, the instantaneous energy accumulation rate and the energy stored reach their final values almost right after the second step. The position of this point is slightly replaced with respect to the corresponding thin plate value, as it is evaluated analytically in Eqs. (38a), (38b). It is also observed that thermal equilibrium is established slightly faster at the plate borders (i.e., at $Y = 0$ and $Y = 1$) than the interior points.

5. Conclusions

The transient conduction of heat through an infinite thin plate was studied when the two free streams surrounding it undergo step changes in their heat transfer coefficients. The changes occur with a time delay. The results show that:

(1) The Biot number step changes, γ_1 and γ_2, define a point Y_0 for which the difference in steady state temperatures (final–initial) is zero. This point separates thermally the plate into two regions and generates opposite thermal stresses.
(2) The additional amount of energy absorbed by the plate in order to create the temperature gradients may well exceed the 10% of the accumulation required to establish the new final steady state. The energy that finally is stored to the plate, $\mathcal{E}_f$, is positive if $Y_0 < 0.5$, negative if $Y_0 > 0.5$ and zero if $Y_0 = 0.5$.
(3) The rate of change of the temperature perturbation during the period between the steps is far larger than the one following the appearance of the second step.
(4) The thermally thick plate leads to large temperature and heat flux perturbations near the time moment that the second step appears.
(5) The recovery time usually increases with the time delay. Under certain conditions ($\tau_d = \tau_d^*$), right after the introduction of the second step the final steady state is reached instantaneously. This corresponds to the shorter recovery time for the entire phenomenon. The specific value (τ_d^*) depends on the Biot number step changes, γ_1 and γ_2, and, consequently, on Y_0.

Appendix A

The initial steady state temperature in the *thin plate* limit, $T_p(t) \equiv T_{0,p}$ at $t = 0$, is obtained from Eq. (30) if we take

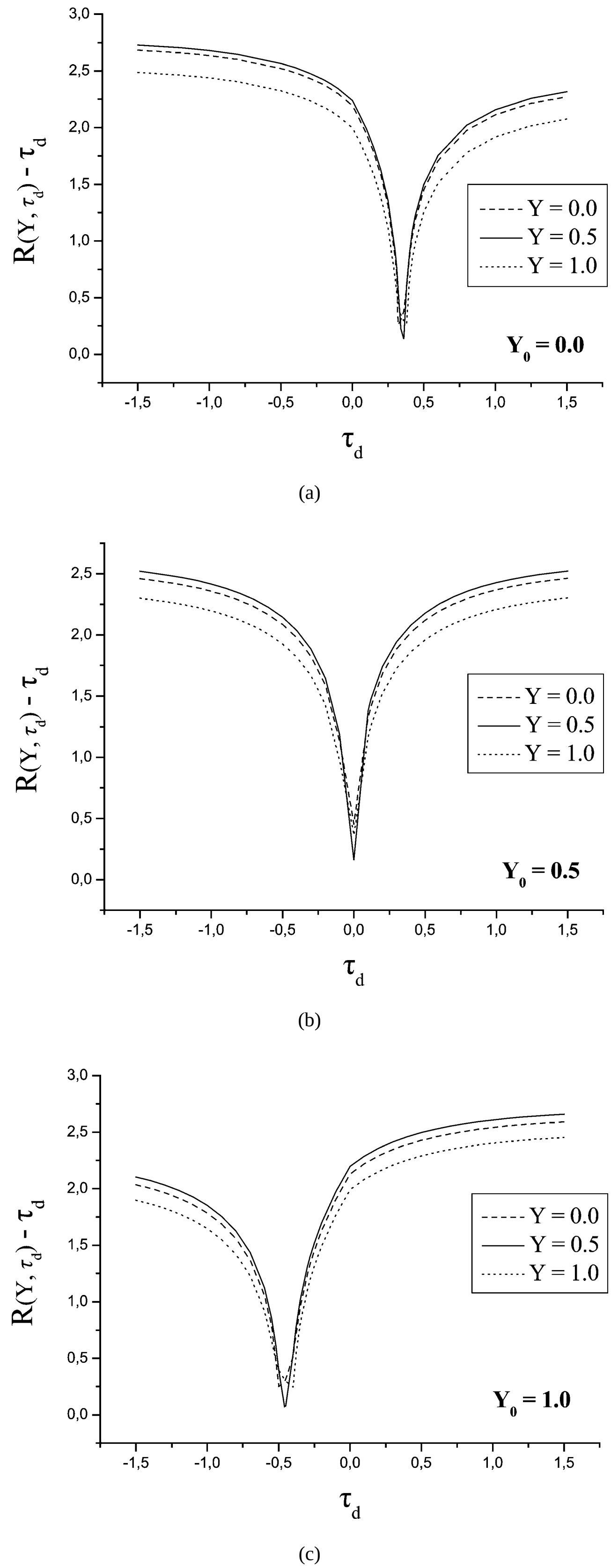


Fig. 7. The combined influence of the delay period and Y_0 on the recovery time when (a) $Y_0 = 0.0$, (b) $Y_0 = 0.5$ and (c) $Y_0 = 1.0$.

into consideration that, initially ($t \leqslant 0$) the plate is in equilibrium with its environment. So, at $t = 0$, we have $\gamma_1 = \gamma_2 = 1$, $dT_{0,p}/dt = 0$ and Eq. (30) yields: $h_{02}(T_{\infty 2} - T_{0,p}) - h_{01}(T_{0,p} - T_{\infty 1}) = 0$. Then, $T_{0,p}$ is obtained as:

$$T_{0,p} = \frac{h_{01}T_{\infty 1} + h_{02}T_{\infty 2}}{h_{01} + h_{02}} = \frac{B_{01}T_{\infty 1} + B_{02}T_{\infty 2}}{B_{01} + B_{02}} \tag{A.1}$$

The non-dimensional thin plate temperature, $\theta_p(\tau)$, is defined as:

$$\theta_p(\tau) = \frac{T_p(t) - T_{0,p}}{T_{\infty 2} - T_{\infty 1}} \tag{A.2}$$

From Eq. (A.2) the following equations result:

$$T_p(t) = (T_{\infty 2} - T_{\infty 1})\theta_p(\tau) + T_{0,p} \tag{A.3}$$

$$\frac{dT_p(t)}{dt} = (T_{\infty 2} - T_{\infty 1})\frac{d\theta_p(\tau)}{dt} = (T_{\infty 2} - T_{\infty 1})\frac{\alpha}{w^2}\frac{d\theta_p(\tau)}{d\tau} \tag{A.4}$$

If we use now Eqs. (A.1), (A.3), (A.4) and take into consideration Eqs. (9) and (11), we obtain the non-dimensional form of Eq. (30), i.e., Eq. (31).

From the definition of the non-dimensional *thick plate* temperature, $\theta(Y, \tau)$, (Eq. (4)), the following equations result:

$$T(y, t) = (T_{\infty 2} - T_{\infty 1})\,\theta(Y, \tau) + T_0(y) \tag{A.5}$$

$$\frac{\partial T(y, t)}{\partial y} = (T_{\infty 2} - T_{\infty 1})\,\frac{\partial \theta(Y, \tau)}{\partial y} + \frac{dT_0(y)}{dy} \tag{A.6}$$

The initial condition $T(y, t) = T_0(y)$ at $t = 0$, is obtained from the solution of $d^2T_0/dy^2 = 0$, which yields $dT_0/dy = const$ and gives the temperature distribution during the initial steady state, i.e., for the period $t \leqslant 0$. In fact, if we consider the boundary conditions (2a) and (2b) at $t = 0$, we have:

$$k\frac{dT_0}{dy} = k\frac{T_0(w) - T_0(0)}{w} = h_{01}\big[T_0(0) - T_{\infty 1}\big] = h_{02}\big[T_{\infty 2} - T_0(w)\big] \tag{A.7}$$

The solution of the system of Eqs. (A.7) yields:

$$T_0(y) = \frac{B_{01}T_{\infty 1} + B_{02}T_{\infty 2} + B_{01}B_{02}T_{\infty 1}}{B_{01} + B_{02} + B_{01}B_{02}} + \frac{B_0}{1 + B_0}(T_{\infty 2} - T_{\infty 1})\frac{y}{w} = \frac{T_{0,p} + B_0T_{\infty 1}}{1 + B_0} + \beta_0(T_{\infty 2} - T_{\infty 1})\frac{y}{w} \tag{A.8}$$

The $T_{0,p}$ involved in Eq. (A.8) is the initial steady state temperature in the thin plate limit, given by Eq. (A.1) while the B_{01}, B_{02}, B_0, β_0 are given by Eqs. (9) and (11).

From Eq. (A.8) it is clear that:

$$\frac{dT_0(y)}{dy} = \frac{\beta_0\,(T_{\infty 2} - T_{\infty 1})}{w} \tag{A.9}$$

$$T_0(y)\big|_{y=0} = T_0(0) = \frac{T_{0,p} + B_0T_{\infty 1}}{1 + B_0} \tag{A.10}$$

$$T_0(y)\big|_{y=w} = T_0(w) = \frac{T_{0,p} + B_0T_{\infty 1}}{1 + B_0} + \beta_0\,(T_{\infty 2} - T_{\infty 1}) \tag{A.11}$$

If we use now Eqs. (A.5), (A.6), (A.9)–(A.11) and take into consideration Eq. (5), we obtain from Eqs. (1), (2a), (2b) and (3), the non-dimensional form of the thick plate equations, i.e., Eqs. (6), (7a), (7b) and (8), respectively.

Appendix B

We have to solve the following equation

$$\frac{\partial^2\theta}{\partial Y^2} = \frac{\partial\theta}{\partial\tau}, \quad 0 < Y < 1 \tag{B.1}$$

with boundary conditions

$$\frac{\partial\theta}{\partial Y} - \gamma_1 Bi_{01}\theta = (\gamma_1 - 1)\beta_0, \quad Y = 0 \tag{B.2a}$$

$$\frac{\partial\theta}{\partial Y} + \gamma_2 Bi_{02}\theta = (\gamma_2 - 1)\beta_0, \quad Y = 1 \tag{B.2b}$$

The initial condition is

$$\theta(Y, \tau) = 0, \quad \tau = 0 \tag{B.3}$$

if the equation is solved in the time interval $0 \leqslant \tau \leqslant \tau_d$, and

$$\theta(Y, \tau) = F(Y), \quad \tau = \tau_d \tag{B.4}$$

if the equation is solved in the time interval $\tau > \tau_d$.

By separating the time and space variables, it is demonstrated easily that the complete solution of the temperature function $\theta(Y, \tau)$ is given in the form

$$\theta(Y, \tau) = \sum_{n=1}^{\infty} c_n\Psi(\lambda_n, Y)\bar{\theta}(\lambda_n, \tau) \tag{B.5}$$

where c_n are unknown coefficients to be determined, $\bar{\theta}(\lambda_n, \tau)$ is the integral transform of $\theta(Y, \tau)$ with respect to the space variable Y in the range $0 \leqslant Y \leqslant 1$, defined as:

$$\bar{\theta}(\lambda_n, \tau) = \int_{Y'=0}^{1} \Psi(\lambda_n, Y')\theta(Y', \tau)\,dY' \tag{B.6}$$

while $\Psi(\lambda_n, Y)$ has the form

$$\Psi(\lambda_n, Y) = \cos\lambda_n Y + \frac{\gamma_1 B_{01}}{\lambda_n}\sin\lambda_n Y \tag{B.7}$$

In fact, $\Psi(\lambda_n, Y)$ is the eigenfunction of the following auxiliary eigenvalue problem

$$\frac{d^2\Psi(Y)}{dY^2} + \lambda^2\Psi(Y) = 0, \quad 0 < Y < 1 \tag{B.8}$$

$$\frac{d\Psi(Y)}{dY} - \gamma_1 B_{01}\Psi(Y) = 0, \quad Y = 0 \tag{B.9a}$$

$$\frac{d\Psi(Y)}{dY} + \gamma_2 B_{02}\Psi(Y) = 0, \quad Y = 1 \tag{B.9b}$$

By applying now the transform (B.6) we take the integral transform of Eq. (B.1)

$$\int_0^1 \Psi(\lambda_n, Y)\frac{\partial^2\theta(Y,\tau)}{\partial Y^2}\,dY = \int_0^1 \Psi(\lambda_n, Y)\frac{\partial\theta(Y,\tau)}{\partial\tau}\,dY = \frac{d\bar{\theta}(\lambda_n,\tau)}{d\tau} \tag{B.10}$$

The term in the first member in Eq. (B.10) is evaluated by making use of the Green's theorem and written as

$$\int_0^1 \Psi(\lambda_n, Y)\frac{\partial^2\theta(Y,\tau)}{\partial Y^2}\,dY = -\lambda_n^2\bar{\theta}(\lambda_n,\tau) - (\gamma_1 - 1)\beta_0\Psi(\lambda_n, Y)\big|_{Y=0} + (\gamma_2 - 1)\beta_0\Psi(\lambda_n, Y)\big|_{Y=1} \tag{B.11}$$

By substituting Eq. (B.11) into Eq. (B.10), the latter it is finally written as:

$$\frac{d\bar{\theta}(\lambda_n,\tau)}{d\tau} + \lambda_n^2\bar{\theta}(\lambda_n,\tau) = A(\lambda_n) \tag{B.12}$$

where we have put for brevity

$$A(\lambda_n) = \beta_0\big[(\gamma_2 - 1)\Psi(\lambda_n, 1) - (\gamma_1 - 1)\Psi(\lambda_n, 0)\big] \tag{B.13}$$

The initial condition for Eq. (B.12) is obtained by taking the integral transform of the initial condition of Eq. (B.1). Let us apply the transform (B.6) to the condition (B.4):

$$\bar{\theta}(\lambda_n,\tau)\big|_{\tau=\tau_d} = \int_{Y'=0}^{1} \Psi(\lambda_n, Y')F(Y')\,dY' \equiv \bar{F}(\lambda_n) \tag{B.14}$$

The solution of Eq. (B.12) subjected to the transformed initial condition (B.14) is:

$$\bar{\theta}(\lambda_n,\tau) = \bar{F}(\lambda_n)e^{-\lambda_n^2(\tau-\tau_d)} + \frac{A(\lambda_n)}{\lambda_n^2}\big[1 - e^{-\lambda_n^2(\tau-\tau_d)}\big] \tag{B.15}$$

In case where the initial condition (B.3) is used, then $\bar{F}(\lambda_n) = 0$ and if we put $\tau_d = 0$ into Eq. (B.15) we obtain

$$\bar{\theta}(\lambda_n,\tau) = \frac{A(\lambda_n)}{\lambda_n^2}\big(1 - e^{-\lambda_n^2\tau}\big) \tag{B.16}$$

The simplest case is when the two-step changes to h_1 and h_2 appear simultaneously. Then $\tau_d = 0$ and, if we introduce (B.16) and (B.7) into the general form of the solution, i.e., Eq. (B.5) and determine the coefficients c_n by taking into consideration the orthogonality of the eigenfunctions $\Psi(\lambda_n, Y)$, we obtain the following solution:

$$\theta(Y,\tau) = 2\beta_0\sum_{n=1}^{\infty} g(\lambda_n)\left(\cos\lambda_n Y + \frac{B_1}{\lambda_n}\sin\lambda_n Y\right) \times \big(1 - e^{-\lambda_n^2\tau}\big) \tag{B.17}$$

where the eigenvalues λ_n are the positive roots of the transcendental equation (42) and $g(\lambda_n)$ are given as:

$$g(\lambda_n) = \frac{(\gamma_2 - 1)\big(\cos\lambda_n + \frac{B_1}{\lambda_n}\sin\lambda_n\big) - (\gamma_1 - 1)}{B_1 + \big(1 + \frac{B_2}{\lambda_n^2 + B_2^2}\big)(\lambda_n^2 + B_1^2)} \tag{B.18}$$

In the general case, the solution depends on the time interval considered:

(a) In the time interval $[0 \leqslant \tau \leqslant \tau_d]$, the initial condition is given by Eq. (B.3). So, the solution is similar to Eq. (B.17), i.e.:

$$\theta(Y,\tau) = 2\beta_0\sum_{m=1}^{\infty} f(\xi_m)\left(\cos\xi_m Y + \frac{\gamma_1 B_{01}}{\xi_m}\sin\xi_m Y\right) \times \big(1 - e^{-\xi_m^2\tau}\big) \tag{B.19}$$

The eigenvalues ξ_m are the positive roots of the transcendental equation (40a) if h_1 changes first, or (40b) if h_2 changes first. For the latter case, in Eq. (B.19) it is set $\gamma_1 = 1$. The functions $f(\xi_m)$ are given in the former case by

$$f(\xi_m) = -\frac{\gamma_1 - 1}{B_1 + \big(1 + \frac{B_{02}}{\xi_m^2 + B_{02}^2}\big)(\xi_m^2 + B_1^2)} \tag{B.20a}$$

and in the latter by

$$f(\xi_m) = \frac{(\gamma_2 - 1)\big(\cos\xi_m + \frac{B_{01}}{\xi_m}\sin\xi_m\big)}{B_{01} + \big(1 + \frac{B_2}{\xi_m^2 + B_2^2}\big)(\xi_m^2 + B_{01}^2)} \tag{B.20b}$$

(b) In the time interval $[\tau > \tau_d]$, the initial condition is given by Eq. (B.4), where $F(Y) = \theta(Y, \tau_d)$ is obtained from Eq. (B.19) for $\tau = \tau_d$. The $\bar{\theta}(\lambda_n, \tau)$, involved in the general form of the solution (i.e., Eq. (B.5)), is given by Eq. (B.15), while $\bar{F}(\lambda_n)$ is obtained from Eq. (B.14) for $F(Y) = \theta(Y, \tau_d)$. The solution is

$$\begin{aligned}\theta(Y,\tau) = {}& 2\beta_0\sum_{n=1}^{\infty} C(\lambda_n) \times \left(\cos\lambda_n Y + \frac{B_1}{\lambda_n}\sin\lambda_n Y\right)e^{-\lambda_n^2(\tau-\tau_d)} \\ & \times \sum_{m=1}^{\infty} f(\xi_m)D(\lambda_n,\xi_m)\big(1 - e^{-\xi_m^2\tau_d}\big) \\ & + 2\beta_0\sum_{n=1}^{\infty} g(\lambda_n)\left(\cos\lambda_n Y + \frac{B_1}{\lambda_n}\sin\lambda_n Y\right) \\ & \times \big(1 - e^{-\lambda_n^2(\tau-\tau_d)}\big)\end{aligned} \tag{B.21}$$

The eigenvalues λ_n are the positive roots of the transcendental equation (42), $f(\xi_m)$ is given by Eqs. (B.20a), (B.20b), $g(\lambda_n)$ is given by Eq. (B.18) and $C(\lambda_n)$ is given as:

$$C(\lambda_n) = \frac{\lambda_n^2}{B_1 + \big(1 + \frac{B_2}{\lambda_n^2 + B_2^2}\big)(\lambda_n^2 + B_1^2)} \tag{B.22}$$

The $D(\lambda_n, \xi_m)$ depends on whether h_1 or h_2 changes first and is given respectively by

$$D(\lambda_n, \xi_m) = \left(1 + \frac{B_1^2}{\lambda_n \xi_m}\right)\frac{\sin(\lambda_n - \xi_m)}{\lambda_n - \xi_m} + \left(1 - \frac{B_1^2}{\lambda_n \xi_m}\right)\frac{\sin(\lambda_n + \xi_m)}{\lambda_n + \xi_m} + \left(\frac{B_1}{\lambda_n} + \frac{B_1}{\xi_m}\right)\left[\frac{2\lambda_n}{\lambda_n^2 - \xi_m^2} - \frac{\cos(\lambda_n - \xi_m)}{\lambda_n - \xi_m} - \frac{\cos(\lambda_n + \xi_m)}{\lambda_n + \xi_m}\right] \tag{B.23a}$$

$$D(\lambda_n, \xi_m) = \left(1 + \frac{B_{01} B_1}{\xi_m \lambda_n}\right)\frac{\sin(\lambda_n - \xi_m)}{\lambda_n - \xi_m} + \left(1 - \frac{B_{01} B_1}{\xi_m \lambda_n}\right)\frac{\sin(\lambda_n + \xi_m)}{\lambda_n + \xi_m} + \left(\frac{B_{01}}{\xi_m} + \frac{B_1}{\lambda_n}\right)\left[\frac{2\lambda_n}{\lambda_n^2 - \xi_m^2} - \frac{\cos(\lambda_n - \xi_m)}{\lambda_n - \xi_m} - \frac{\cos(\lambda_n + \xi_m)}{\lambda_n + \xi_m}\right] \tag{B.23b}$$

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